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Dissipativity of Volterra functional differential equations ☆

Liping Wen *, Shoufu Li

School of Mathematics and Computational Science, Xiangtan University, Xiangtan, Hunan 411105, PR China

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Abstract

This paper is concerned with the dissipativity of Volterra functional differential equations in a Hilbert space. A sufficient condition for dissipativity of one class of such equations is obtained. This result is applied to delay differential equations and integro-differential equations to obtain dissipativity results that are more general and deeper than related results in the previous literature.

Keywords: Volterra functional differential equations; Dynamical systems; Dissipativity

1. Introduction

Let H be a real or complex Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and corresponding norm $\|\cdot\|$. Let X be a dense continuously imbedded subspace of H. For any given closed interval $I \subset R$, let the symbol $C_X(I)$ denote the Banach space consisting of all continuous mappings $x : I \to X$ with norm $\|x\|_\infty = \max_{t\in I} \|x(t)\|$. Consider the Volterra functional differential equations (VFDEs) initial value problem

$$\begin{cases} y'(t) = f(t, y(t), y(\cdot)), & t \geqslant 0, \\ y(t) = \varphi(t), & -\tau \leqslant t \leqslant 0, \end{cases} \tag{1.1}$$

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* Corresponding author.
E-mail address: lpwen@xtu.edu.cn (L. Wen).

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where τ is a positive constant, $\varphi \in C_X[-\tau, 0]$ is a given initial function, $f:[0,+\infty) \times X \times C_X[-\tau,+\infty] \to H$ is a given locally Lipschitz continuous mapping satisfying

$$2\Re\langle u, f\big(t, u, \psi(\cdot)\big)\rangle \leqslant \gamma(t) + \alpha(t)\|u\|^2 + \beta(t) \max_{t-\mu_2(t)\leqslant\xi\leqslant t-\mu_1(t)} \big\|\psi(t)\big\|^2,$$
$$u \in X,\ \psi \in C_X[-\tau,+\infty],\ t \in [0,+\infty), \tag{1.2}$$

where $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are bounded continuous functions on the interval $[0,+\infty)$, $\mu_1(t)$ and $\mu_2(t)$ satisfy

$$0 < \inf_{0\leqslant\xi<+\infty} \mu_1(t) \leqslant \mu_1(t) \leqslant \mu_2(t) \leqslant t+\tau \quad \forall t \in [0,+\infty) \tag{1.3}$$

and

$$\lim_{t\to+\infty}\big(t-\mu_2(t)\big) = +\infty. \tag{1.4}$$

Throughout this paper, we define

$$\alpha_0 = \sup_{0\leqslant t<+\infty} \alpha(t), \qquad \beta_0 = \sup_{0\leqslant t<+\infty} \beta(t), \qquad \gamma_0 = \sup_{0\leqslant t<+\infty} \gamma(t), \tag{1.5a}$$

$$\mu_1^{(0)} = \inf_{0\leqslant t<+\infty} \mu_1(t),$$
$$\mu_2^{(0)}(\xi_1,\xi_2) = \inf_{\xi_1\leqslant t\leqslant\xi_2}\big(t-\mu_2(t)\big) \quad \forall \xi_1,\xi_2\colon 0\leqslant\xi_1\leqslant\xi_2<+\infty, \tag{1.5b}$$

and we always assume that $f(t,\psi(t),\psi(\cdot))$ is independent of the values of the function $\psi(\xi)$ with $t<\xi<+\infty$, i.e. $f(t,\psi(t),\psi(\cdot))$ is a Volterra functional and that problem (1.1) has a true solution $y(t) \in C_X[-\tau,+\infty]$. Note that condition (1.2) arises in the study of several frequently analyzed partial differential equations, including the Navier–Stokes equation whose semi-discrete approximation can be written in form (1.1) with $\beta(t) \equiv 0$ (cf. [1]).

In this paper, we concentrate on the dissipativity of problem (1.1) in VFDEs. In Section 2 a sufficient condition for the dissipativity of the problem is obtained. In Section 3, applying the sufficient condition to some special cases, several dissipativity results for delay differential equations and integro-differential equations, respectively, are obtained, which are more general and deeper than the existing related results in literature (cf. [5,9]).

Many interesting problems in physics and engineering are modelled by dissipative systems in VFDEs, which are characterized by possessing a bounded absorbing set that all trajectories enter in a finite time and thereafter remain inside (cf. [1,2]). Therefore, our results obtained in this paper are of importance in theory and practice.

2. Main results and their proofs

Proposition 2.1. *If the mapping f in problem* (1.1) *satisfies condition* (1.2)*, then we have*

$$\gamma(t) \geqslant 0, \quad \beta(t) \geqslant 0 \quad \forall t \in [0,+\infty). \tag{2.1}$$

Definition 2.2. Problem (1.1) in VFDEs is said to be dissipative in H if there exists a bounded set $B \subset H$, such that for any given bounded set $\Phi \subset H$, there is a time $t_0 = t_0(\Phi)$, such that for any given initial function $\varphi \in C_X[-\tau,0]$ with $\varphi(t)$ contained in Φ for all $t \in [-\tau,0]$, the values of the corresponding solution $y(t)$ of the problem are contained in B for all $t \geqslant t_0$. Here B is called an absorbing set of the problem.

Lemma 2.3. *Suppose $y(t)$ is a solution of problem* (1.1) *satisfying condition* (1.2). *Then we have*

$$\begin{aligned}\|y(t_2)\|^2 &\leqslant \exp\left(\int_{t_1}^{t_2} \alpha(\eta)\,d\eta\right)\|y(t_1)\|^2 + \int_{t_1}^{t_2} \gamma(\zeta)\exp\left(\int_{\zeta}^{t_2}\alpha(\eta)\,d\eta\right)d\zeta \\ &\quad + \int_{t_1}^{t_2} \beta(\zeta)\exp\left(\int_{\zeta}^{t_2}\alpha(\eta)\,d\eta\right)d\zeta \max_{\mu_2^{(0)}(t_1,t_2)\leqslant\xi\leqslant t_2-\mu_1^{(0)}}\|y(\xi)\|^2 \\ &\forall t_1,t_2\colon\ 0\leqslant t_1\leqslant t_2<+\infty.\end{aligned}\tag{2.2}$$

Proof. Define

$$Y(t)=\|y(t)\|^2=\langle y(t),y(t)\rangle \tag{2.3}$$

and

$$Q(t)=p(t)\big(Y(t)+\delta q(t)+s(t)\big),\quad t_1\leqslant t\leqslant t_2, \tag{2.4}$$

where

$$p(t)=\exp\left(-\int_0^t \alpha(\eta)\,d\eta\right),\qquad q(t)=-\big[p(t)\big]^{-1}\int_0^t \beta(\zeta)p(\zeta)\,d\zeta,$$

$$s(t)=-\big[p(t)\big]^{-1}\int_0^t \gamma(\zeta)p(\zeta)\,d\zeta, \tag{2.5}$$

δ is a constant to be determined. Then we have

$$p'(t)=-\alpha(t)p(t),\qquad q'(t)=\alpha(t)q(t)-\beta(t),\qquad s'(t)=\alpha(t)s(t)-\gamma(t),$$

and therefore together with (1.2)

$$\begin{aligned}Q'(t) &= p(t)\big[-\alpha(t)\big(Y(t)+\delta q(t)+s(t)\big)+2\Re\langle y(t),y'(t)\rangle+\delta\big(\alpha(t)q(t)-\beta(t)\big) \\ &\quad +\alpha(t)s(t)-\gamma(t)\big] \\ &\leqslant p(t)\beta(t)\Big(\max_{t-\mu_2(t)\leqslant\xi\leqslant t-\mu_1(t)}\|Y(\xi)\|-\delta\Big).\end{aligned}\tag{2.6}$$

Choose

$$\delta=\max_{\mu_2^{(0)}(t_1,t_2)\leqslant\xi\leqslant t_2-\mu_1^{(0)}} Y(\xi)$$

and note $\beta(t)\geqslant 0$ by Proposition 2.1. Then (2.6) leads to $Q'(t)\leqslant 0$ which means that $Q(t)$ is a non-increased function on the interval $[t_1,t_2]$. Therefore inequality (2.2) follows. This completes the proof of Lemma 2.3. □

Theorem 2.4. *Suppose that $y(t)$ is a solution of problem* (1.1) *satisfying condition* (1.2), *and that $\alpha_0+\beta_0<0$. Then,*

(i) *for any given $\varepsilon > 0$, there exists a positive number $t^* = t^*(\|\varphi\|_\infty, \varepsilon)$, such that*

$$\|y(t)\|^2 < \frac{\gamma_0}{-(\alpha_0+\beta_0)} + \varepsilon \quad \forall t > t^*; \tag{2.7}$$

(ii) *for any given $\varepsilon > 0$, problem* (1.1) *is dissipative with an absorbing set*

$$B = B\big(0, \sqrt{-\gamma_0/(\alpha_0+\beta_0)+\varepsilon}\big).$$

Proof. For any given $t \in (0, \mu_1^{(0)}]$, we choose $t_1 = 0$, $t_2 = t$. Using Lemma 2.3 and noting Proposition 2.1, we thus get

$$\begin{aligned}
\|y(t)\|^2 &\leqslant \exp\left(\int_0^t \alpha(\eta)\,d\eta\right)\|y(0)\|^2 + \int_0^t \gamma(\zeta)\exp\left(\int_\zeta^t \alpha(\eta)\,d\eta\right)d\zeta \\
&\quad + \int_0^t \beta(\zeta)\exp\left(\int_\zeta^t \alpha(\eta)\,d\eta\right)d\zeta \max_{\mu_2^{(0)}(0,t)\leqslant\xi\leqslant t-\mu_1^{(0)}} \|y(\xi)\|^2 \\
&\leqslant \left(\exp\left(\int_0^t \alpha(\eta)\,d\eta\right) + \int_0^t \beta(\zeta)\exp\left(\int_\zeta^t \alpha(\eta)\,d\eta\right)d\zeta\right)\|\varphi\|_\infty^2 \\
&\quad + \int_0^t \gamma(\zeta)\exp\left(\int_\zeta^t \alpha(\eta)\,d\eta\right)d\zeta \\
&\leqslant \left(1 + \int_0^t \big(\alpha(\zeta)+\beta(\zeta)\big)\exp\left(\int_\zeta^t \alpha(\eta)\,d\eta\right)d\zeta\right)\|\varphi\|_\infty^2 + \left(1 - e^{\alpha_0\mu_1^{(0)}}\right)\frac{\gamma_0}{-\alpha_0}.
\end{aligned} \tag{2.8}$$

Note that here

$$1 + \int_0^t \big(\alpha(\zeta)+\beta(\zeta)\big)\exp\left(\int_\zeta^t \alpha(\eta)\,d\eta\right)d\zeta \in (0,1).$$

It follows directly from (2.8) that

$$\|y(t)\|^2 \leqslant \|\varphi\|_\infty^2 + r, \quad 0 < t \leqslant \mu_1^{(0)}, \tag{2.9}$$

where

$$r = \left(1 - e^{\alpha_0\mu_1^{(0)}}\right)\frac{\gamma_0}{-\alpha_0}.$$

Furthermore, for any given $t > \mu_1^{(0)}$, we choose $t_1 = t - \mu_1^{(0)}$, $t_2 = t$. Then from Lemma 2.3 we get

$$\|y(t)\|^2 \leqslant \exp\left(\int_{t-\mu_1^{(0)}}^t \alpha(\eta)\,d\eta\right)\big\|y\big(t-\mu_1^{(0)}\big)\big\|^2 + \int_{t-\mu_1^{(0)}}^t \gamma(\zeta)\exp\left(\int_\zeta^t \alpha(\eta)\,d\eta\right)d\zeta$$

$$+\int_{t-\mu_1^{(0)}}^{t} \beta(\zeta)\exp\left(\int_{\zeta}^{t}\alpha(\eta)\,d\eta\right)d\zeta \max_{\mu_2^{(0)}(t-\mu_1^{(0)},t)\leqslant\xi\leqslant t-\mu_1^{(0)}}\left\|y(\xi)\right\|^2$$
$$\leqslant\left[e^{\alpha_0\mu_1^{(0)}}+\frac{\beta_0}{-\alpha_0}\left(1-e^{\alpha_0\mu_1^{(0)}}\right)\right]\max_{\mu_2^{(0)}(t-\mu_1^{(0)},t)\leqslant\xi\leqslant t-\mu_1^{(0)}}\left\|y(\xi)\right\|^2+r,$$

which gives

$$\left\|y(t)\right\|^2\leqslant\theta\max_{\mu_2^{(0)}(t-\mu_1^{(0)},t)\leqslant\xi\leqslant t-\mu_1^{(0)}}\left\|y(\xi)\right\|^2+r\leqslant\theta\max_{-\tau\leqslant\xi\leqslant t-\mu_1^{(0)}}\left\|y(\xi)\right\|^2+r,\quad t>\mu_1^{(0)},\tag{2.10}$$

where

$$\theta=\frac{\beta_0}{-\alpha_0}+\frac{\alpha_0+\beta_0}{\alpha_0}e^{\alpha_0\mu_1^{(0)}}.\tag{2.11}$$

We now consider the following two cases successively.

Case 1. $\|\varphi\|_\infty^2\leqslant\theta(\|\varphi\|_\infty^2+r)$. It can be proved by mathematical induction that

$$\left\|y(t)\right\|^2\leqslant\theta^n\|\varphi\|_\infty^2+r\sum_{j=0}^{n}\theta^j,\quad t\in\left(n\mu_1^{(0)},(n+1)\mu_1^{(0)}\right],\ n=1,2,\ldots.\tag{2.12}$$

In fact, when $n=1$, from (2.10) and (2.9) we have

$$\left\|y(t)\right\|^2\leqslant\theta\max_{-\tau\leqslant\xi\leqslant\mu_1^{(0)}}\left\|y(\xi)\right\|^2+r\leqslant\theta\max\left\{\|\varphi\|_\infty^2+r,\|\varphi\|_\infty^2\right\}+r=\theta\left(\|\varphi\|_\infty^2+r\right)+r,$$

which means (2.12) holds when $n=1$. If (2.12) holds for $n<l$, where l is a positive integer, then, for $t\in(l\mu_1^{(0)},(l+1)\mu_1^{(0)}]$, it follows from (2.10) that

$$\begin{aligned}\left\|y(t)\right\|^2&\leqslant\theta\max_{-\tau\leqslant\xi\leqslant l\mu_1^{(0)}}\left\|y(\xi)\right\|^2+r\\&=\theta\max\left\{\max_{1\leqslant k\leqslant l}\max_{(k-1)\mu_1^{(0)}\leqslant t\leqslant k\mu_1^{(0)}}\left\|y(t)\right\|^2,\max_{t\in[-\tau,0]}\left\|y(t)\right\|^2\right\}+r\\&\leqslant\theta\max\left\{\max_{0\leqslant k\leqslant l-1}\left(\theta^k\|\varphi\|_\infty^2+r\sum_{j=0}^{k}\theta^j\right),\|\varphi\|_\infty^2\right\}+r.\end{aligned}\tag{2.13}$$

On the other hand, when $\|\varphi\|_\infty^2\leqslant\theta(\|\varphi\|_\infty^2+r)$ we have

$$\theta^k\|\varphi\|_\infty^2+r\sum_{j=0}^{k}\theta^j=\theta^{k-1}\left(\theta\left(\|\varphi\|_\infty^2+r\right)\right)+r\sum_{j=0}^{k-1}\theta^j\geqslant\theta^{k-1}\|\varphi\|_\infty^2+r\sum_{j=0}^{k-1}\theta^j.$$

Therefore, from (2.13) we obtain

$$\|y(t)\|^2 \leqslant \theta\left(\theta^{l-1}\|\varphi\|_\infty^2 + r\sum_{j=0}^{l-1}\theta^j\right) + r = \theta^l\|\varphi\|_\infty^2 + r\sum_{j=0}^{l}\theta^j,$$

which shows that (2.12) holds for any $n \geqslant 1$.

Case 2. $\|\varphi\|_\infty^2 > \theta(\|\varphi\|_\infty^2 + r)$. At first, from (1.4), we can construct a strictly increasing sequence $\{\xi_k\}$ ($\xi_0 = 0$) satisfying

(i) $\lim_{k\to\infty}\xi_k = +\infty$; (2.14)

(ii) $t - \mu_2(t) > \xi_k,\ \forall t > \xi_{k+1},\ k = 0, 1, \ldots;$ (2.15)

(iii) there exits a strictly increasing positive integer sequence $\{n_k\}$ such that $\lim_{k\to\infty} n_k = \infty$, and that

$$\xi_k = n_k\mu_1^{(0)}, \quad k = 1, 2, \ldots. \tag{2.16}$$

In fact, assume that ξ_k and n_k have been chosen appropriately where $k \geqslant 0$. Since $\lim_{t\to+\infty}(t - \mu_2(t)) = +\infty$, there exists a constant $M > \xi_k$ such that $t - \mu_2(t) > \xi_k$ for all $t > M$. If $M + 1$ is just an integral times of $\mu_1^{(0)}$, e.g. there exists a positive integer number i such that $M + 1 = i\mu_1^{(0)}$, then we let $\xi_{k+1} = M + 1$ and $n_{k+1} = i$, otherwise there exists a natural number j consequentially satisfying $j\mu_1^{(0)} < M + 1 < (j + 1)\mu_1^{(0)}$, we let $\xi_{k+1} = (j + 1)\mu_1^{(0)}$ and $n_{k+1} = j + 1$. Thus we obtain the sequences $\{\xi_k\}$ and $\{n_k\}$ which satisfy our requests.

From (2.10), we can obtain that

$$\|y(t)\|^2 \leqslant \|\varphi\|_\infty^2 + r, \quad t \in (0, \xi_1]. \tag{2.17}$$

In fact, when $t \in (0, \mu_1^{(0)}]$, (2.17) holds obviously. Now we assume that (2.17) holds for $t \in \bigcup_{i=1}^{k}((i - 1)\mu_1^{(0)}, i\mu_1^{(0)}]$, where $1 \leqslant k < n_1$. Then, for $t \in (k\mu_1^{(0)}, (k + 1)\mu_1^{(0)}]$, it follows from (2.10) that

$$\|y(t)\|^2 \leqslant \theta \max_{-\tau\leqslant\xi\leqslant k\mu_1^{(0)}}\|y(\xi)\|^2 + r \leqslant \theta\big(\|\varphi\|_\infty^2 + r\big) + r \leqslant \|\varphi\|_\infty^2 + r,$$

where we have used the condition $\|\varphi\|_\infty^2 > \theta(\|\varphi\|_\infty^2 + r)$. This shows that (2.17) holds for any $t \in (0, \xi_1]$.

Furthermore, we also can prove the following inequality by mathematical induction

$$\|y(t)\|^2 \leqslant \theta^n\|\varphi\|_\infty^2 + r\sum_{j=0}^{n}\theta^j, \quad t \in (\xi_n, \xi_{n+1}],\ n = 0, 1, 2, \ldots. \tag{2.18}$$

In fact, (2.17) shows that (2.18) obviously holds for $n = 0$. Now we assume that (2.18) holds for any $n < l$, where l is a positive integer. In the following we prove by mathematical induction that the inequality

$$\max_{\xi_l<\xi\leqslant t-\mu_1^{(0)}}\|y(\xi)\|^2 \leqslant \theta^l\|\varphi\|_\infty^2 + r\sum_{j=0}^{l}\theta^j, \quad t \in (\xi_l, \xi_{l+1}], \tag{2.19}$$

holds.

When $t \in (\xi_l, \xi_l + \mu_1^{(0)}]$ it follows from (2.10) and (2.15) that

$$
\begin{aligned}
\|y(t)\|^2 &\leqslant \theta \max_{\mu_2^{(0)}(t-\mu_1^{(0)},t)\leqslant \xi \leqslant t-\mu_1^{(0)}} \|y(\xi)\|^2 + r \leqslant \theta \max_{\xi_{l-1}\leqslant \xi \leqslant \xi_l} \|y(\xi)\|^2 + r \\
&\leqslant \theta\left(\theta^{l-1}\|\varphi\|_\infty^2 + r\sum_{j=0}^{l-1}\theta^j\right) + r = \theta^l\|\varphi\|_\infty^2 + r\sum_{j=0}^{l}\theta^j.
\end{aligned}
\tag{2.20}
$$

Now we assume that the inequality

$$
\|y(t)\|^2 \leqslant \theta^l\|\varphi\|_\infty^2 + r\sum_{j=0}^{l}\theta^j
\tag{2.21}
$$

is satisfied with $t \in \bigcup_{i=0}^{k-1}(\xi_l + i\mu_1^{(0)}, \xi_l + (i+1)\mu_1^{(0)}]$, where $1 \leqslant k < n_{l+1} - n_l$. Then when $t \in (\xi_l + k\mu_1^{(0)}, \xi_l + (k+1)\mu_1^{(0)}]$ it follows from (2.10) and (2.15) that

$$
\begin{aligned}
\|y(t)\|^2 &\leqslant \theta \max_{\mu_2^{(0)}(t-\mu_1^{(0)},t)\leqslant \xi \leqslant t-\mu_1^{(0)}} \|y(\xi)\|^2 + r \leqslant \theta \max_{\xi_{l-1}<\xi \leqslant t-\mu_1^{(0)}} \|y(\xi)\|^2 + r \\
&= \theta \max\Big\{\max_{\xi_{l-1}<\xi\leqslant\xi_l}\|y(\xi)\|^2, \max_{\xi_l<\xi\leqslant t-\mu_1^{(0)}}\|y(\xi)\|^2\Big\} + r \\
&= \theta \max\Big\{\max_{\xi_{l-1}<\xi\leqslant\xi_l}\|y(\xi)\|^2, \max_{0\leqslant i\leqslant k-1}\max_{\xi_l+i\mu_1^{(0)}<\xi\leqslant\xi_l+(i+1)\mu_1^{(0)}}\|y(\xi)\|^2\Big\} + r \\
&\leqslant \theta \max\left\{\theta^{l-1}\|\varphi\|_\infty^2 + r\sum_{j=0}^{l-1}\theta^j, \theta^l\|\varphi\|_\infty^2 + r\sum_{j=0}^{l}\theta^j\right\} + r \\
&= \theta^l\|\varphi\|_\infty^2 + r\sum_{j=0}^{l}\theta^j,
\end{aligned}
\tag{2.22}
$$

where we have used that

$$
\theta^l\|\varphi\|_\infty^2 + r\sum_{j=0}^{l}\theta^j = \theta^{l-1}\big(\theta\big(\|\varphi\|_\infty^2 + r\big)\big) + r\sum_{j=0}^{l-1}\theta^j \leqslant \theta^{l-1}\|\varphi\|_\infty^2 + r\sum_{j=0}^{l-1}\theta^j.
$$

(2.20) and (2.22) show that (2.19) holds. Thus from (2.19) we obtain that (2.18) holds for $n = l$.

A combination of (2.12) and (2.18) shows that the open ball $B = B(0, \sqrt{-\gamma_0/(\alpha_0+\beta_0)+\varepsilon})$ is an absorbing set for any $\varepsilon > 0$, and that the system is dissipative, which completes the proof of Theorem 2.4. □

3. Application to some special cases

3.1. Application to delay differential equations

Consider the initial value problem in delay differential equations

$$
\begin{cases}
y'(t) = g(t, y(t), y(\eta_1(t)), y(\eta_2(t)), \ldots, y(\eta_r(t))), & t \in [0, +\infty), \\
y(t) = \varphi(t), & t \in [-\tau, 0],
\end{cases}
\tag{3.1a}
$$

where each $\eta_i \in C_R[0, +\infty)$ satisfies

$$-\tau \leqslant \eta_i(t) \leqslant t - \mu_1^{(0)} \quad \forall t \in [0, +\infty),\ i = 1, \dots, r, \tag{3.1b}$$

with constant $\mu_1^{(0)} > 0$. The mapping $g : [0, +\infty) \times X^{r+1} \to X$ satisfying

$$2\Re\big\langle u, g(t, u, x_1, x_2, \dots, x_r)\big\rangle \leqslant \gamma(t) + \alpha(t)\|u\|^2 + \beta(t) \max_{1\leqslant i\leqslant r} \|x_i\|^2$$
$$\forall t \in [0, +\infty),\ u, x_1, x_2, \dots, x_r \in X, \tag{3.2}$$

and the meaning of other symbols are the same as mentioned in Section 2. We always assume that problem (3.1) has a true solution $y(t) \in C_X[-\tau, +\infty)$.

Let

$$\hat f\big(t, u, \psi(\cdot)\big) = g\big(t, u, \psi\big(\eta_1(t)\big), \psi\big(\eta_2(t)\big), \dots, \psi\big(\eta_r(t)\big)\big)$$
$$\forall t \in [0, +\infty),\ u \in X,\ \psi \in C_X[-\tau, +\infty).$$

Then problem (3.1) satisfying condition (3.2) can be equivalently written in the pattern of the initial value problem (1.1) in VFDE, i.e.

$$\begin{cases} y'(t) = \hat f(t, y(t), y(\cdot)), & 0 \leqslant t < +\infty, \\ y(t) = \varphi(t), & -\tau \leqslant t \leqslant 0, \end{cases} \tag{3.3}$$

where the mapping $\hat f : [0, +\infty) \times X \times C_X[-\tau, +\infty) \to X$ satisfies the condition

$$\begin{aligned} 2\Re\big\langle u, \hat f\big(t, u, \psi(\cdot)\big)\big\rangle &= 2\Re\big\langle u, g\big(t, u, \psi\big(\eta_1(t)\big), \dots, \psi\big(\eta_r(t)\big)\big)\big\rangle \\ &\leqslant \gamma(t) + \alpha(t)\|u\|^2 + \beta(t) \max_{1\leqslant i\leqslant r} \big\|\psi\big(\eta_i(t)\big)\big\|^2 \\ &\leqslant \gamma(t) + \alpha(t)\|u\|^2 + \beta(t) \max_{t-\mu_2(t)\leqslant \xi\leqslant t-\mu_1(t)} \big\|\psi(\xi)\big\|^2 \end{aligned}$$
$$\forall t \in [0, +\infty),\ u \in X,\ \psi \in C_X[-\tau, +\infty)$$

with

$$\mu_1(t) = t - \max_{1\leqslant i\leqslant r} \eta_i(t), \qquad \mu_2(t) = t - \min_{1\leqslant i\leqslant r} \eta_i(t).$$

Therefore, Theorem 2.4 in the present paper can be directly applied to this special case, and we thus obtain the following result.

Theorem 3.1. *Suppose that $y(t)$ is a solution of problem* (3.1) *satisfying condition* (3.2) *and*

$$\lim_{t\to+\infty}\Big(\min_{1\leqslant i\leqslant r} \eta_i(t)\Big) = +\infty,$$

and that $\alpha_0 + \beta_0 < 0$. Then,

(i) *for any given $\varepsilon > 0$, there exists a positive number $t^* = t^*(\|\varphi\|_\infty, \varepsilon)$, such that*

$$\big\|y(t)\big\|^2 < \frac{\gamma_0}{-(\alpha_0 + \beta_0)} + \varepsilon \quad \forall t > t^*;$$

(ii) *for any given $\varepsilon > 0$, problem* (3.1) *is dissipative with an absorbing set*

$$B = B\big(0, \sqrt{-\gamma_0/(\alpha_0 + \beta_0) + \varepsilon}\big).$$

3.2. Application to integro-differential equations

Consider the initial value problem in integro-differential equations

$$\begin{cases} y'(t) = g\big(t, y(t), \int_{\eta_1(t)}^{\eta_2(t)} K(t,\xi,y(\xi))\,d\xi\big), & t \in [0,+\infty), \\ y(t) = \varphi(t), & t \in [-\tau, 0], \end{cases} \tag{3.4a}$$

where $\eta_1, \eta_2 \in C_R[0,+\infty)$ satisfy

$$-\tau \leqslant \eta_1(t) \leqslant \eta_2(t) \leqslant t - \mu_1^{(0)} \quad \forall t \in [0,+\infty), \tag{3.4b}$$

with constant $\mu_1^{(0)} > 0$, the continuous mapping $K : [0,+\infty) \times [-\tau,+\infty) \times X \to X$ satisfies

$$\big\|K(t,\xi,x)\big\| \leqslant L_K \|x\| \quad \forall t \in [0,+\infty),\ \xi \in [-\tau,+\infty),\ x \in X, \tag{3.5}$$

with constant $L_K > 0$, and continuous mapping $g : [0,+\infty) \times X \times X \to X$ satisfies

$$2\Re\big\langle u, g(t,u,x)\big\rangle \leqslant \gamma(t) + \alpha(t)\|u\|^2 + \tilde{\beta}(t)\|x\|^2 \quad \forall t \in [0,+\infty),\ u, x \in X. \tag{3.6}$$

The meaning of other symbols are the same as mentioned in Section 2, and we always assume that problem (3.4) has a true solution $y(t) \in C_X[-\tau,+\infty)$.

Let

$$\tilde{f}\big(t,u,\psi(\cdot)\big) = g\left(t, u, \int_{\eta_1(t)}^{\eta_2(t)} K\big(t,\xi,\psi(\xi)\big)\,d\xi\right)$$
$$\forall t \in [0,+\infty),\ u \in X,\ \psi \in C_X[-\tau,+\infty).$$

Then the mapping $\tilde{f} : [0,+\infty) \times X \times C_X[-\tau,+\infty) \to X$ satisfies

$$\begin{aligned} 2\Re\big\langle u, \tilde{f}\big(t,u,\psi(\cdot)\big)\big\rangle &= 2\Re\left\langle u, g\left(t, u, \int_{\eta_1(t)}^{\eta_2(t)} K\big(t,\xi,\psi(\xi)\big)\,d\xi\right)\right\rangle \\ &\leqslant \gamma(t) + \alpha(t)\|u\|^2 + \tilde{\beta}(t)\left\|\int_{\eta_1(t)}^{\eta_2(t)} K\big(t,\xi,\psi(\xi)\big)\,d\xi\right\|^2 \end{aligned}$$
$$\forall t \in [0,+\infty),\ u \in X,\ \psi \in C_X[-\tau,+\infty).$$

Because

$$\begin{aligned} \left\|\int_{\eta_1(t)}^{\eta_2(t)} K\big(t,\xi,\psi(\xi)\big)\,d\xi\right\| &\leqslant \int_{\eta_1(t)}^{\eta_2(t)} \big\|K\big(t,\xi,\psi(\xi)\big)\big\|\,d\xi \leqslant L_K \int_{\eta_1(t)}^{\eta_2(t)} \big\|\psi(\xi)\big\|\,d\xi \\ &\leqslant L_K\big(\eta_2(t) - \eta_1(t)\big) \max_{\eta_1(t) \leqslant \xi \leqslant \eta_2(t)} \big\|\psi(\xi)\big\|. \end{aligned}$$

Therefore, problem (3.4) satisfying conditions (3.5) and (3.6) can be equivalently written in the pattern of the initial value problem (1.1) in VFDE, i.e.

$$\begin{cases} y'(t) = \tilde{f}(t, y(t), y(\cdot)), & 0 \leqslant t < +\infty, \\ y(t) = \varphi(t), & -\tau \leqslant t \leqslant 0, \end{cases} \tag{3.7}$$

with

$$\mu_1(t) = t - \eta_2(t), \qquad \mu_2(t) = t - \eta_1(t), \qquad \beta(t) = \tilde{\beta}(t)\big(L_K\big(\eta_2(t) - \eta_1(t)\big)\big)^2.$$

Thus the result for VFDEs obtained in the present paper can be directly applied to this special case, and we thus obtain the following dissipativity result.

Theorem 3.2. *Suppose that $y(t)$ is a solution of problem* (3.4) *satisfying conditions* (3.5), (3.6) *with $\alpha_0 + \beta_0 < 0$ and*

$$\lim_{t\to+\infty} \eta_1(t) = +\infty.$$

Then,

(i) *for any given $\varepsilon > 0$, there exists a positive number $t^* = t^*(\|\varphi\|_\infty, \varepsilon)$, such that*

$$\big\|y(t)\big\|^2 < \frac{\gamma_0}{-(\alpha_0 + \beta_0)} + \varepsilon \quad \forall t > t^*;$$

(ii) *for any given $\varepsilon > 0$, problem* (3.4) *is dissipative with an absorbing set*

$$B = B\big(0, \sqrt{-\gamma_0/(\alpha_0 + \beta_0) + \varepsilon}\,\big).$$

3.3. Comparison with the existing results

(i) In 1994, Humphries and Stuart [2] studied the dissipativity for initial value problems in ordinary differential equations in C^m, and obtained that the systems

$$\begin{cases} y'(t) = f(y(t)), & t \geqslant 0, \\ y(t) = y_0, \end{cases} \tag{3.8}$$

with f satisfying

$$2\Re\big\langle u, f(u)\big\rangle \leqslant \gamma + \alpha\|u\|^2, \quad u \in C^m, \tag{3.9}$$

are dissipative. They also studied the dissipativity of Runge–Kutta methods applying to (3.8). In 1997, Hill [3,4] extended the mentioned results above to systems (3.8) in a Hilbert space.

(ii) In 2000, Huang [5] further investigated the dissipativity of the system in delay differential equations

$$\begin{cases} y'(t) = f(y(t), y(t-\tau)), & t \geqslant 0, \\ y(t) = \varphi(t), & -\tau \leqslant t \leqslant 0, \end{cases} \tag{3.10}$$

with a positive delay term τ, where $\varphi(t)$ is a continuous function, $f : X \times X \to H$ is a locally Lipschitz continuous function which satisfies the following structural assumptions

$$2\Re\big\langle u, f(u, v)\big\rangle \leqslant \gamma + \alpha\|u\|^2 + \beta\|v\|^2, \quad u, v \in X, \tag{3.11}$$

where α, β and γ are real constants. He pointed out that system (3.10) satisfying (3.11) is dissipative if only $\alpha + \beta < 0$. He also obtained the dissipativity results of numerical methods which are applied to (3.10) such as Runge–Kutta methods [5], one-lag methods [6], θ-methods [7] (also see [10]) and multistep Runge–Kutta methods [8].

Since Theorem 3.1 of the present paper is also available for the case of variable delay, the result of Theorem 3.1 is more general and deeper than that obtained by Huang [5] mentioned above.

(iii) In 2004, Tian [9] studied the dissipativity of non-linear system in delay differential equations with a bounded variable lag

$$\begin{cases} y'(t) = f(t, y(t), y(t-\tau(t))), & t \geqslant 0, \\ y(t) = \varphi(t), & -\tau \leqslant t \leqslant 0, \end{cases} \tag{3.12}$$

where y and f are p-vector-valued functions, the lag function $\tau(t)$ satisfies

$$0 \leqslant \tau(t) \leqslant \tau$$

with a constant $\tau \geqslant 0$, f satisfies the following assumption

$$\Re\langle u, f(t,u,v)\rangle \leqslant \gamma(t) + \alpha(t)\|u\|^2 + \beta(t)\|v\|^2 \quad \forall t \in [0,\infty),\ u, v \in C^p, \tag{3.13}$$

with the continuous functions $\alpha(t)$, $\beta(t)$, $\gamma(t)$ satisfy for $t \geqslant 0$ $\exists \alpha_0 > 0$, $0 \leqslant q < 1$, $\gamma^* > 0$:

$$0 \leqslant \gamma(t) < \gamma^*, \qquad \alpha(t) \leqslant -\alpha_0 < 0, \qquad 0 \leqslant \beta(t) \leqslant -q\alpha(t)$$

and verified that system (3.12) is dissipative. He also obtained the numerical dissipativity of θ-methods, which extended the dissipativity results of DDEs in literature further.

Since Theorem 3.1 of the present paper do not require the delay to be bounded, at this point, the result of Theorem 3.1 is more general than that obtained by Tian [9] mentioned above.

(iv) So far we have not seen in literature any dissipativity results similar to that of Theorem 3.2 for integro-differential equations.

References

[1] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Appl. Math. Sci. Ser., vol. 68, Springer, Berlin, 1988.

[2] A.R. Humphries, A.M. Stuart, Runge–Kutta methods for dissipative and gradient dynamical systems, SIAM J. Numer. Anal. 31 (1994) 1452–1485.

[3] A.T. Hill, Global dissipativity for A-stable methods, SIAM J. Numer. Anal. 34 (1997) 119–142.

[4] A.T. Hill, Dissipativity of Runge–Kutta methods in Hilbert spaces, BIT 37 (1997) 37–42.

[5] C.M. Huang, Dissipativity of Runge–Kutta methods for dynamical systems with delays, IMA J. Numer. Anal. 20 (2000) 153–166.

[6] C.M. Huang, Dissipativity of one-lag methods for dynamical systems with delays, Appl. Numer. Math. 35 (2000) 11–22.

[7] C.M. Huang, G.N. Chen, Dissipativity of linear θ-methods for delay dynamical systems, Math. Numer. Sin. 22 (2001) 501–506 (in Chinese).

[8] C.M. Huang, Q.S. Chang, Dissipativity of multistep Runge–Kutta methods for dynamical systems with delays, Math. Comput. Modelling 40 (2004) 1285–1296.

[9] H.J. Tian, Numerical and analytic dissipativity of the θ-method for delay differential equation with a bounded variable lag, Internat. J. Bifur. Chaos 14 (2004) 1839–1845.

[10] L.Q. Fan, Y.Y. Zhang, J.X. Xiang, H.J. Tian, Numerical dissipativity of two-stage θ-method for delay differential equations, J. Syst. Simulation 17 (2005) 599–600.